SENSITIVITY ANALYSIS FOR THE 2D NAVIER-STOKES EQUATIONS WITH APPLICATIONS TO CONTINUOUS DATA ASSIMILATION

Abstract. We rigorously prove the well-posedness of the formal sensitivity equations with respect to the viscosity corresponding to the 2D incompressible Navier-Stokes equations. Moreover, we do so by showing a sequence of difference quotients converges to the unique solution of the sensitivity equations for both the 2D Navier-Stokes equations and the related data assimilation equations, which utilize the continuous data assimilation algorithm proposed by Azouani, Olson, and Titi. As a result, this method of proof provides uniform bounds on difference quotients, demonstrating parameter recovery algorithms that change parameters as the system evolves will be well-behaved. Furthermore, our analysis can be extended to analyze the sensitivity of the 2D Euler equations to a viscous regularization. We also note that this appears to be the first such rigorous proof of global existence and uniqueness to strong or weak solutions to the sensitivity equations for the 2D Navier-Stokes equations (in the natural case of zero initial data), and that they can be obtained as a limit of difference quotients with respect to the viscosity.

1. Introduction

Turbulent flows are well-known to be chaotic, in the sense that the solutions to mathematical models of such flows are highly sensitive to initial conditions (see, e.g., [17, 40]). However, sensitivity with respect to physical parameters is also an important consideration in terms of making reliably accurate predictions. Parameter sensitivity is often measured by formally considering the derivative of a solution with respect to a particular parameter; however, the only rigorous justification of this approach in the literature seems to be limited to linear equations, or non-linear equations under assumptions on the nonlinearity which are too strong to include, e.g., the Navier-Stokes equations of fluids (see, e.g., [13, 37] for a semigroup theory approach). Therefore, in the present work, we provide a fully rigorous proof of the global well-posedness of the sensitivity equations for the 2D Navier-Stokes equations. Specifically, we give a rigorous proof of the existence of unique weak and strong solutions with zero[1] initial data to the associated viscosity sensitivity equations specifically for the 2D Navier-Stokes equations. Moreover, we prove that the derivative of solutions with respect to the viscosity is a limit of difference quotients corresponding to different viscosities.

We also extend our results to the case of a data assimilation algorithm. This is because the motivation for this present work arose from our recent work [14], where an algorithm was proposed to recover an unknown viscosity, or equivalently Reynolds number. This algorithm works in tandem with a data assimilation method proposed in [5, 4]. This algorithm, commonly referred to as the Azouani-Olson-Titi (AOT) or Continuous Data Assimilation (CDA) algorithm, has seen much recent work (see, e.g., [1, 2, 6, 7, 8, 9, 14, 15, 16, 21, 52, 20, 22, 23, 24, 25, 26, 27, 28, 29, 32, 33, 34, 35, 36, 38, 42, 43, 44, 45, 49, 50, 51, 53, 54, 56, 61, 64, 72] and the references therein.) Specifically, [4] considers the 2D Navier-Stokes system, written abstractly in the form

$$\frac{du}{dt} = F_\nu(u).$$

The difficulty is that the initial data is unknown; however, it is assumed that the solution can be measured at certain points. In order to converge to the correct solution, it is proposed to instead

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[1]Note that considerations of sensitivity arise in the context of perturbations; hence, the natural initial data for a sensitivity equation is identically zero data.

consider the system

$$\begin{cases} \dfrac{dv}{dt} = F_\nu(v) + \mu(I_h(u) - I_h(v)) \\ v(0) = v_0, \end{cases}$$

where $\mu > 0$ is a sufficiently large positive relaxation parameter, $I_h(u)$ represents the observational measurements with sufficiently small spacing $h > 0$, v_0 is arbitrarily chosen in a certain function space, and F_ν is a nonlinear, nonlocal differential operator depending on the viscosity parameter $\nu > 0$. I_h is an interpolation operator satisfying certain bounds (see Section 2); e.g., I_h could be piecewise linear interpolation at given nodes. In [4], it was proven that v converges to u exponentially fast in time in certain standard norms. Later, [14] investigated the case of an unknown viscosity, gave estimates for the resulting error in the solution, proposed an algorithm to recover the unknown viscosity, and demonstrated computationally that the algorithm converges exponentially fast in time to the correct solution. However, the algorithm in [14] introduces a discontinuous change in the viscosity during the simulation, leading to a desire to ensure that this abrupt change does not lead to undesirable behavior. Hence, we also prove that the difference quotient methods developed here can be used to prove rigorous results for the sensitivity equations of the modified system of equations via the data assimilation algorithm. For this system, we prove that the derivative of solutions with respect to the viscosity is a well-defined object which is bounded in appropriate function spaces; additionally we prove that the corresponding sensitivity equations are globally well-posed in time in an appropriate sense and that strong solutions are unique.

Furthermore, one can also perceive the importance of analyzing the sensitivity of a system of equations to a parameter-based regularization. Hence, we also prove rigorously that the sensitivity equations for a viscous regularization (i.e., a Navier-Stokes-like perturbation) of the 2D incompressible Euler equations in 2D (under periodic boundary conditions) have a unique strong solution that can be realized as a sequence of difference quotients.

Sensitivity for partial differential equations has been studied formally in many contexts; see, e.g., [3, 11, 12, 13, 19, 30, 37, 41, 47, 48, 57, 58, 59, 60, 63, 67, 71, 72]. In [67], it was argued, though only formally, that the sensitivity equations for the steady-state 2D Navier-Stokes equations are globally well-posed. Some analysis for the sensitivity equations has been carried out in the slightly more general context of a large eddy simulation (LES) model of the 2D Navier-Stokes equations in an unpublished PhD thesis [59], where a formal argument for the global existence and uniqueness of the equations was given, based on formal energy estimates.

The paper is organized as follows. In Section 2, we describe the mathematical framework for the problems we consider. In Section 3 we prove the global existence and uniqueness of solutions to the sensitivity equations. Moreover, we show that these solutions can be realized as limits of difference quotients. In Section 4, we extend the results in the previous section to the context of AOT data assimilation algorithm. Finally, we summarize our results and implications of this work in Section 5.

2. Preliminaries

In this section, we lay out notation and recall some standard results about the incompressible Navier-Stokes equations. Proofs can be found in, e.g., [18, 31, 65, 70, 69]. Similarly, equivalent results for the modified data assimilation equations given by the AOT algorithm are stated without proof as well, since proofs were given in [4]. On a general open spatial domain Ω, the incompressible Navier-Stokes equations are given by

$$u_t + (u\cdot\nabla)u = -\nabla p + \nu\triangle u + f, \qquad \text{in } \Omega\times[0,T], \tag{2.1a}$$

$$\nabla\cdot u = 0, \qquad \text{in } \Omega\times[0,T], \tag{2.1b}$$

$$u(x,0) = u_0(x), \qquad \text{in } \Omega. \tag{2.1c}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity, p is the (density normalized) pressure, and f is a (density normalized) body force. In this paper, we take Ω to be the torus, i.e. $\Omega = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which is an open, bounded, and connected domain with empty boundary. We denote the space

$$\mathcal{V} := \{f : \Omega \to \mathbb{R}^2 \mid f \in \dot{C}_p^\infty(\mathbb{T}^2)\},$$

where $\dot{C}_p^\infty(\mathbb{T}^2)$ is the set of all mean-free, periodic, and infinitely differentiable functions on $\mathbb{T}^2$. Denote by H the closure of $\mathcal{V}$ in $L^2(\Omega;\mathbb{R}^2)$ and by V the closure of $\mathcal{V}$ in $H^1(\Omega;\mathbb{R}^2)$. Since H and V are closed subspaces of $L^2(\Omega;\mathbb{R}^2)$ and $H^1(\Omega;\mathbb{R}^2)$, respectively, they are Hilbert spaces which inherit inner products denoted by

$$(u,v) = \int_{\mathbb{T}^2} u \cdot v \, dx \qquad ((u,v)) = \sum_{i,j=1}^{2} \int_{\mathbb{T}^2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, dx,$$

with the corresponding norms denoted by $|u| = \sqrt{(u,u)}$ and $\|u\| = \sqrt{((u,u))}$. We denote by $L^p(0,T;X)$ those functions which are $L^p((0,T))$ in time with values in the Banach space X in the sense of the Bochner integral; similarly for expressions such as $C^k([0,T];X)$. We denote the dual space of V by V'.

We consider the equivalent problem applying the Leray projection to (2.1), where the Leray projection onto divergence-free vector fields is defined by $P_\sigma u := u - \nabla \triangle^{-1} \nabla \cdot u$, $P_\sigma : L^2(\Omega) \to H$. As in [4], we denote the Stokes operator $A : \mathcal{D}(A) \to H$, where $\mathcal{D}(A) := \{u \in V : Au \in H\}$ is the domain of A, and the bilinear term $B : V \times V \to V'$ as the continuous extensions of the operators A, defined on $\mathcal{V}$, and B, defined on $\mathcal{V} \times \mathcal{V}$, by

$$Au = -P_\sigma \triangle u \qquad \text{and} \qquad B(u,v) = P_\sigma(u \cdot \nabla v).$$

Due to boundedness of the domain and the mean-zero condition, the following Poincaré inequalities hold:

$$\lambda_1 \|u\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 \quad \text{for} \quad u \in V, \tag{2.2}$$

$$\lambda_1 \|\nabla u\|_{L^2}^2 \le \|Au\|_{L^2}^2 \quad \text{for} \quad u \in \mathcal{D}(A). \tag{2.3}$$

We note that, as proven in, e.g., [18, 65, 70], A is a linear self-adjoint positive definite operator with a compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_i\}_{i\in\mathbb{N}}$ in V such that $Aw_i = \lambda_i w_i$, where the corresponding eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ are positive and monotonically increasing. Hence, for $s \in \mathbb{R}$, the fractional operators A^s may be defined via $A^s w_i = \lambda_i^2 w_i$ and extension to appropriate spaces by linearity and density.

As proven in, e.g., [18, 65, 70], the bilinear operator, B, has the property

$$\langle B(u,v), w \rangle_{V',V} = - \langle B(u,w), v \rangle_{V',V}, \tag{2.4}$$

for all $u, v, w \in V$, which directly implies that

$$\langle B(u,w), w \rangle_{V',V} = 0, \tag{2.5}$$

for all $u, v, w \in V$. Furthermore, we have the following inequalities:

$$|\langle B(u,v), w \rangle| \le \|u\|_{L^\infty(\Omega)} \|v\| |w| \qquad \text{for } u \in L^\infty(\Omega), v \in V, w \in H \tag{2.6}$$

$$|\langle B(u,v), w \rangle| \le c|u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2} \qquad \text{for } u, v, w \in V, \tag{2.7}$$

$$|(B(u,v), w)| \le c|u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w| \qquad \text{for } u \in V, v \in \mathcal{D}(A), w \in H \tag{2.8}$$

$$|(B(u,v), w)| \le c|u|^{1/2} |Au|^{1/2} \|v\| |w| \qquad \text{for } u \in \mathcal{D}(A), v \in V, w \in H. \tag{2.9}$$

Due to the periodic boundary conditions, it also holds (in 2D) that

$$(B(w,w), Aw) = 0 \quad \text{for every} \quad w \in \mathcal{D}(A). \tag{2.10}$$

From this an some manipulation, it follows that for $u, w \in \mathcal{D}(A)$,

$$(B(u,w), Aw) + (B(w,u), Aw) = -(B(w,w), Au). \tag{2.11}$$

Additionally, further properties of the bilinear term are stated in Lemma 2.1, which we prove using similar strategies as in [65, 70].

Lemma 2.1. *Suppose* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{b_n\}_{n\in\mathbb{N}}$ *are uniformly bounded sequences in* $L^2(0,T;V)\cap L^\infty(0,T;H)$*. Then* $\|B(a_n,b_n)\|_{L^2(0,T;V')}$ *is uniformly bounded in* n*. Moreover, if* $\{a_n\}_{n\in\mathbb{N}}$ *and* $\{b_n\}_{n\in\mathbb{N}}$ *are uniformly bounded in* $L^2(0,T;\mathcal{D}(A))\cap L^\infty(0,T;V)$*, then* $\|B(a_n,b_n)\|_{L^2(0,T;H)}$ *is uniformly bounded in* n*.*

Proof. By the definition of the dual norm, (2.4), and (2.7)

$$\|B(a_n,b_n)\|_{V'} = \sup_{\substack{w\in V\\ \|w\|=1}} |(B(a_n,w),b_n)| \le k|a_n|^{1/2}\|a_n\|^{1/2}|b_n|^{1/2}\|b_n\|^{1/2}.$$

Using Hölder's inequality,

$$\begin{aligned}\|B(a_n,b_n)\|^2_{L^2(0,T;V')} &\le \int_0^T k^2|a_n|\|a_n\||b_n|\|b_n\|ds\\ &\le k^2\|a_n\|_{L^\infty(0,T;H)}\|b_n\|_{L^\infty(0,T;H)}\|a_n\|_{L^2(0,T;V)}\|b_n\|_{L^2(0,T;V)}.\end{aligned}$$

Hence, since $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^2(0,T;V)\cap L^\infty(0,T;H)$, it follows that $\|B(a_n,b_n)\|_{L^2(0,T;V')}$ is uniformly bounded in n.

Next, suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded uniformly in $L^2(0,T;\mathcal{D}(A))\cap L^\infty(0,T;V)$. Then by definition of the dual norm, (2.8), and (2.2),

$$\begin{aligned}\|B(a_n,b_n)\|^2_{L^2(0,T;H)} &\equiv \int_0^T |B(a_n,b_n)|^2dt \le c\int_0^T |a_n|\|a_n\|\|b_n\||Ab_n|dt\\ &\le \frac{c}{\lambda_1}\|a_n\|^2_{L^\infty(0,T;V)}\int_0^T |Ab_n|^2dt = \frac{c}{\lambda_1}\|a_n\|^2_{L^\infty(0,T;V)}\|b_n\|^2_{L^2(0,T;\mathcal{D}(A))},\end{aligned}$$

which implies that $\|B(a_n,b_n)\|_{L^2(0,T;H)}$ is uniformly bounded in n. □

Finally, without loss of generality, we make the assumption that $f\in L^\infty(0,T;H)$ so that $P_\sigma f = f$. This allows us to apply P_σ to (2.1) to obtain the equivalent set of equations

$$\frac{d}{dt}u + B(u,u) = -\nu Au + f, \qquad \text{in } \Omega\times[0,T], \tag{2.12a}$$

$$u(x,0) = u_0(x), \qquad \text{in } \Omega. \tag{2.12b}$$

Using the following corollary of de Rham's theorem [70, 31]

$$g = \nabla p \text{ with } p \text{ a distribution if and only if } \langle g,h\rangle = 0 \text{ for all h} \in \mathcal{V}, \tag{2.13}$$

one can recover the pressure term. Furthermore, we will utilize the following lemma due to Lions and Magenes as stated in [70].

Lemma 2.2. [70] *Let* V,H,V' *be three Hilbert spaces such that* $V\subset H\equiv H'\subset V'$*, where* V' *is the dual of* V*. If a function* $u\in L^2(0,T;V)$ *and its time derivative* $u'\in L^2(0,T;V')$ *then* u *is almost everywhere equal to a function continuous from* $[0,T]$ *into* H *and the following equality holds in the scalar distribution sense on (0,T):*

$$\frac{d}{dt}|u|^2 = 2\langle u',u\rangle.$$

For reference, we state the standard definition of weak and strong solutions to (2.12).

Definition 2.3. Let $T>0$. Let initial data $u_0\in H$ and forcing $f\in L^\infty(0,\infty;V')$. A *weak solution* of (2.1) is an element $u\in L^2(0,T;V)\cap C([0,T];H)$ satisfying $\frac{du}{dt}\in L^2(0,T;V')$ and

$$\left\langle \frac{d}{dt}u,\phi\right\rangle + \langle B(u,u),\phi\rangle + \nu\langle Au,\phi\rangle = 0 \tag{2.14}$$

for a.e. $t \in [0,T]$ and for all $\phi \in V$. Furthermore, the map $u_0 \in H \mapsto u \in H$ is continuous for a.e. $t \in [0,T]$, i.e. the initial data is satisfied in the sense of $C([0,T];H)$.

If, in addition, $f \in L^\infty(0,\infty;H)$ and $u_0 \in V$, then a *strong solution* of (2.1) is defined to be a weak solution such that $u \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ and $\frac{d\widetilde{u}}{dt} \in L^2(0,T;H)$, and moreover u satisfies (2.14) for a.e. $t \in [0,T]$ and for all $\phi \in H$. Furthermore, the map $u_0 \in V \mapsto u \in V$ is continuous for a.e. $t \in [0,T]$, i.e. the initial data is satisfied in the sense of $C([0,T];V)$.

It is well-established that given a force $f \in L^2(0,\infty;H^{-1}$ and initial data $u_0 \in H$, a unique weak solution to (2.12) exists globally in time (see, e.g., [18, 31, 65, 70]). For many real world applications, it is important to consider the sensitivity of (2.1) to the parameters since it is not necessarily the case we have an exact estimate on the said parameters; however, additional uncertainty in modeling real world systems is introduced by the fact that we do not expect to know u_0 exactly, and so cannot compute $u(t)$ from (2.12). Hence, we will also analyze the sensitivity equations corresponding to a modified system of equations that utilizes measured data collected on the true field $u(t)$ over the time interval $[0,T]$. This modified system of equations incorporates the measured data by introducing a feedback control involving the interpolated data $I_h(u(t))$ into (2.12), resulting in the following system which was proposed and studied in [4, 5] (see also [10]).

$$v_t + (v\cdot\nabla)v = -\nabla q + \nu_{\text{appx}}\triangle v + f + \mu(I_h(u) - I_h(v)) \tag{2.15a}$$

$$v(x,0) = v_0(x), \tag{2.15b}$$

or, with the Leray projection applied,

$$\frac{d}{dt}v + B(v,v) = -\nu_{\text{appx}}Av + f + \mu P_\sigma(I_h(u) - I_h(v)) \tag{2.16a}$$

$$v(x,0) = v_0(x). \tag{2.16b}$$

Here, $\nu_{\text{appx}} > 0$ is a kinematic viscosity approximating $\nu > 0$, $\mu > 0$ is a positive relaxation parameter, and I_h is a linear operator satisfying

$$\|\varphi - I_h(\varphi)\|^2_{L^2(\Omega)} \le c_0 h^2 \|\varphi\|^2_{H^1(\Omega)} \tag{2.17}$$

Assuming either no-slip Dirichlet or periodic boundary conditions (and allowing for even more general interpolation operators I_h), [4] proved that (2.16) has a unique solution, stated in the following theorem.

Theorem 2.4. *Suppose I_h satisfies* (2.17) *and $\mu c_0 h^2 \le \nu_{appx}$, where c_0 is the constant from* (2.17). *Then the continuous data assimilation equations* (2.16) *possess a unique solution v that satisfies*

$$v \in C([0,T];V) \cap L^2((0,T);\mathcal{D}(A)) \text{ and } \frac{dv}{dt} \in L^2((0,T);H), \tag{2.18}$$

for any $T > 0$. Furthermore, the map $u_0 \in V \mapsto u \in V$ is continuous for a.e. $t \in [0,T]$, i.e. the initial data is satisfied in the sense of $C([0,T];V)$.

For equations (2.12) and (2.16), we denote the dimensionless Grashof numbers as

$$G_1 = \frac{1}{4\pi^2\nu^2} \limsup_{t\to\infty} \|f(t)\|_{L^2(\Omega)} \tag{2.19}$$

$$G_2 = \frac{1}{4\pi^2\nu^2_{\text{appx}}} \limsup_{t\to\infty} \|f(t)\|_{L^2(\Omega)}. \tag{2.20}$$

Finally, we consider the Euler equations in 2D without forcing:

$$u_t + u\cdot\nabla u + \nabla p = 0, \tag{2.21a}$$

$$\nabla\cdot u = 0 \tag{2.21b}$$

$$u(x,0) = u_0. \tag{2.21c}$$

Applying the Leray projection yields the system

$$\frac{du}{dt} + B(u,u) = 0, \tag{2.22a}$$

$$u(x,0) = u_0. \tag{2.22b}$$

The relevant literature on the inviscid limit of solutions to (2.12) to sufficiently smooth solutions of Euler is usually considered in the whole space $\mathbb{R}^2$ [46, 68, 39, 62, 18], but we utilize the relevant theorems straightforwardly extendable to the periodic setting (see, e.g., [55]). We state the types of solutions to Euler that are limits of solutions to (2.1).

Theorem 2.5. *(Masmoudi, 2007* [55]*) Let $s > 0$ and suppose $u \in C([0,T);\mathcal{D}(A^{s/2}))$ is a solution of the Euler system* (2.22) *for some $T > 0$ with $u_0 \in$. Let $u_0^n \in \mathcal{D}(A^{s/2})$ such that $u_0^n \to u_0$ in $\mathcal{D}(A^{s/2})$ as $n \to \infty$. Then, for all $0 < T_0 < T$, for any sequence $\nu_n \to 0$, the corresponding unique sequence of solutions $u^n \in C([0,T_0];\mathcal{D}(A^{s/2}))$ to* (2.1) *with $f \equiv 0$ is such that*

$$\|u^n - u\|_{L^\infty(0,T_0;\mathcal{D}(A^{s/2}))} \to 0,\ n \to \infty.$$

For simplicity, we take $f \equiv 0$ for a direct application of this theorem from [55]. We also note that since we are in the periodic domain $\mathbb{T}^2$ instead of $\mathbb{R}^2$, this implies that convergence of $u_n \to u$ also holds in $L^\infty(0,T_0;\mathcal{D}(A^{s'/2}))$ for all $s' \leq s$.

3. Sensitivity for 2D Navier-Stokes & The Inviscid Limit

In this section, we analyze the sensitivity of w to the viscosity by considering individually the sensitivity of u and v to the viscosity. We wish to consider taking a derivative of equations (2.12a) and (2.16a) with respect to the viscosity. This has been done formally in many works on sensitivity (see, e.g., [3, 11, 12, 19, 30, 41, 47, 48, 59, 60, 71]), yielding what are known as the *sensitivity equations*. However, to the best of our knowledge, a rigorous treatment has yet to appear in the literature. Therefore, we provide a rigorous justification here of the existence and uniqueness of weak and strong solutions to the sensitivity equations in the case of zero initial data, which is the natural data for the sensitivity equation, as discussed below. Moreover, we prove that these solutions can be realized as limits of difference quotients of Navier-Stokes solutions with respect to different viscosities. Indeed, this is the method of our existence proofs, rather than using, e.g., Galerkin methods, fixed-point methods, etc. Proofs using limits of difference quotients have appeared in the literature before, such as in standard proofs of elliptic regularity, the corresponding result for the Stokes equations, etc. However, in the present context (i.e., the time-dependent sensitivity equations for 2D Navier-Stokes), we believe such a proof strategy is novel.

Working formally for a moment, we take the derivative of (2.1) with respect to ν, and denote (again, formally) $\widetilde{u} := \frac{du}{d\nu}$ and $\widetilde{p} := \frac{dp}{d\nu}$, to obtain

$$\widetilde{u}_t + \widetilde{u} \cdot \nabla u + u \cdot \nabla \widetilde{u} - \nu \triangle \widetilde{u} - \triangle u + \nabla \widetilde{p} = 0, \tag{3.1a}$$

$$\nabla \cdot \widetilde{u} = 0. \tag{3.1b}$$

These are known as the sensitivity equations for the Navier-Stokes equations. Similarly we formally take the derivative of (2.15) with respect to ν_{appx}, denoting $\widetilde{v} := \frac{dv}{d\nu_{\text{appx}}}$ and $\widetilde{q} := \frac{dq}{d\nu_{\text{appx}}}$,

$$\widetilde{v}_t + \widetilde{v} \cdot \nabla v + v \cdot \nabla \widetilde{v} - \nu_{\text{appx}} \triangle \widetilde{v} - \triangle v + \nabla \widetilde{q} = \mu I_h(\widetilde{u} - \widetilde{v}), \tag{3.2a}$$

$$\nabla \cdot \widetilde{v} = 0. \tag{3.2b}$$

Below, we prove some well-posedness results for these systems in the case of zero initial data. We begin by defining what we mean by solutions.

Remark 3.1. The following proofs follow mostly standard techniques; however, they establish that the solutions of the sensitivity equations can indeed be realized as limits of difference quotients, which is the first time this has been done rigorously. Moreover, we note that the analysis for the sensitivity of the Euler equations with respect to viscous perturbations is done via methods typically used in Navier-Stokes-type analysis rather than methods used in the analysis of the Euler equations. Furthermore, this method of proof highlights the independence of the initial data on the parameter, and specifies what one should mean when referring to the sensitivity equations.

Definition 3.2. Let $T > 0$. Let $u \in L^2(0,T;V) \cap C([0,T];H)$ be a weak solution to (2.1). A *weak solution* of (3.1) is a weak solution in the sense of Definition 2.3, but with the equation instead given by

$$\left\langle \frac{d}{dt}\widetilde{u}, \phi \right\rangle + \langle B(\widetilde{u}, u), \phi\rangle + \langle B(u, \widetilde{u}), \phi\rangle + \nu \langle A\widetilde{u}, \phi\rangle + \langle Au, \phi\rangle = 0. \tag{3.3}$$

If, in addition $\widetilde{u}_0 \in V$, and u is a strong solution to (2.1), then we similarly define a *strong solution* of (3.1) in the same sense as Definition 2.3 with respect to the above equation.

As discussed in Remark 3.6 below, we only give a definition of strong solutions for the assimilation equations to avoid redundancy.

Definition 3.3. Let $T > 0$. Let v be a strong solution to (2.16) with initial data $v_0 \in V$ and forcing $f \in L^\infty(0,\infty;H)$. A strong solution of (3.2) is a strong solution in the sense of Definition 2.3 with the equation given by

$$\left\langle \frac{d}{dt}\widetilde{v}, \phi \right\rangle + \langle B(\widetilde{v}, v), \phi\rangle + \langle B(v, \widetilde{v}), \phi\rangle + \nu_{\text{appx}} \langle A\widetilde{v}, \phi\rangle + \langle Av, \phi\rangle = \mu \langle I_h(\widetilde{u} - \widetilde{v}), \phi\rangle .$$

Before we prove the existence and uniqueness of solutions with zero initial data to these equations, we first consider equations for the difference quotients. Note that, since these are simple arithmetic operations on the Navier-Stokes equations, the manipulations can be performed rigorously, not just formally. To this end, let (u_1, p_1) be a strong solution to (2.1) with viscosity ν_1 and (u_2, p_2) be a strong solution to (2.1) with viscosity ν_2 with the same initial data. We take the difference of the two versions of (2.1), each with viscosities ν_1 and ν_2. We then divide by the difference in viscosities, yielding the system

$$w_t + u_2 \cdot \nabla w + w \cdot \nabla u_1 - \nu_2 \triangle w - \triangle u_1 + \nabla P = 0, \tag{3.4a}$$

$$\nabla \cdot w = 0, \tag{3.4b}$$

$$w(x, 0) = 0, \tag{3.4c}$$

where $w = \frac{u_1 - u_2}{\nu_1 - \nu_2}$ and $P := \frac{p_1 - p_2}{\nu_1 - \nu_2}$. As defined, w is a strong solution to (3.4), and note that $u_1 = (\nu_1 - \nu_2)w + u_2$. Additionally, $w \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ and $\frac{dw}{dt} \in L^2(0,T;H)$. However, we need to establish that w is the unique solution to (3.4), which is the content of Lemma 3.4 below.

Lemma 3.4. *Let $T > 0$ be given, and let u_1, $u_2 \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ be strong solutions to* (2.16)*, with viscosities ν_1 and ν_2, respectively. There exists one and only one solution w to* (3.4) *that lies in $L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$, i.e. for all $\phi \in H$,*

$$\left(\frac{d}{dt} w, \phi \right) + (B(w, u_1), \phi) + (B(u_2, w), \phi) + \nu_2 (Aw, \phi) + (Au_1, \phi) = 0,$$

where $\frac{d}{dt} w \in L^2(0,T;H)$. Moreover, the map $u_0 \in V \mapsto u \in V$ is continuous for a.e. $t \in [0,T]$, i.e. the solution depends continuously on the initial data.

Next, we consider difference quotients for the assimilation system (2.16). Let (v_1, q_1) be the strong solution to (2.16) with viscosity $\nu_{\text{appx},1}$ and (v_2, q_2) be the strong solution to (2.16) with viscosity $\nu_{\text{appx},2}$. Subtracting the two equations and dividing by the difference in the viscosities yields

$$\widetilde{w}_t + \widetilde{w} \cdot \nabla v_1 + v_2 \cdot \nabla \widetilde{w} - \nu_{\text{appx},2} \triangle \widetilde{w} - \triangle v_1 + \nabla Q = \mu I_h (w - \widetilde{w}) \tag{3.5a}$$

$$\nabla \cdot \widetilde{w} = 0 \tag{3.5b}$$

$$\widetilde{w}(x, 0) = 0, \tag{3.5c}$$

where $\widetilde{w} := \frac{v_1 - v_2}{\nu_{\text{appx},1} - \nu_{\text{appx},2}}$ and $Q := \frac{q_1 - q_2}{\nu_{\text{appx},1} - \nu_{\text{appx},2}}$. It follows from the definition that $\widetilde{w}$ is a strong solution to (3.5). Moreover, note that $v_1 = (\nu_{\text{appx},1} - \nu_{\text{appx},2})\widetilde{w} + v_2$. Additionally, $\widetilde{w} \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ and $\frac{d}{dt}\widetilde{w} \in L^2(0,T;H)$.

Lemma 3.5. *Let $T > 0$ be given, and let v_1, $v_2 \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ be strong solutions to* (2.16)*, with viscosities $\nu_{appx,1}$ and $\nu_{appx,2}$, respectively. There exists a unique strong solution w to* (3.5) *that lies in $L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$, in the sense that for all $\phi \in H$,*

$$\left(\frac{d}{dt}\widetilde{w}, \phi\right) + (B(v_2, \widetilde{w}), \phi) + (B(\widetilde{w}, \nabla v_1), \phi) + \nu_{appx,2}(A\widetilde{w}, \phi) + (Av_1, \phi) = \mu(P_\sigma I_h(w - \widetilde{w}), \phi),$$

where $\frac{d}{dt}\widetilde{w} \in L^2(0,T;H)$. Moreover, the map $u_0 \in V \mapsto u \in V$ is continuous for a.e. $t \in [0,T]$, i.e. the solution depends continuously on the initial data.

Remark 3.6. The proofs of the above two lemmata are very similar; hence, we only present the proof of Lemma 3.5. Moreover, we also note that in the case $\mu = 0$, the proof of Lemma 3.4 holds *mutatis mutandis* in the case where u_1, $u_2 \in C([0,T];H) \cap L^2(0,T;V)$ are only assumed to be weak solutions to the 2D Navier-Stokes equations, and then one obtains uniqueness of weak solutions to (3.4) in the class $C([0,T];H) \cap L^2(0,T;V)$. However, in the case $\mu > 0$, the notion of weak solutions for the assimilation equations (2.16) has not been established in the literature for general interpolants I_h, and therefore we assume that the solutions v_1 and v_2 are strong solutions to (2.16), and prove the uniqueness of strong solutions to (3.4). The existence of weak solutions is attainable if further assumptions are made on I_h (e.g., if I_h is a projection onto low Fourier modes) but we do not pursue this here as we want to focus on general interpolants.

Proof. Suppose there exist two solutions $\widetilde{w}_1$ and $\widetilde{w}_2$. We consider the difference of the equations

$$\frac{d}{dt}\widetilde{w}_1 + B(\widetilde{w}_1, v_1) + B(v_2, \widetilde{w}_1) + \nu_{\text{appx},2}A\widetilde{w}_1 + Av_1 = \mu P_\sigma I_h(w - \widetilde{w}_1) \tag{3.6}$$

and

$$\frac{d}{dt}\widetilde{w}_2 + B(\widetilde{w}_2, v_1) + B(v_2, \widetilde{w}_2) + \nu_{\text{appx},2}A\widetilde{w}_2 + Av_1 = \mu P_\sigma I_h(w - \widetilde{w}_2). \tag{3.7}$$

Namely, denoting $W := \widetilde{w}_1 - \widetilde{w}_2$, yields

$$\frac{d}{dt}W + B(W, v_1) + B(v_2, W) + \nu_{\text{appx},2}AW = -\mu P_\sigma I_h(W) \tag{3.8}$$

with $W(0) = 0$. So, W must be a solution to the above equation. Taking the inner product with AW, one can follow the argument in [4] line-by-line to obtain to obtain that

$$||W(t)||^2 \le \|W(0)\|^2 e^{ct} \tag{3.9}$$

for some constant $c > 0$. Since $W(0) = 0$, this implies that $W \equiv 0$. Hence, solutions to (3.5) are unique. Moreover, (3.9) gives the continuous dependence on initial data. □

Since systems (3.4) and (3.5) have unique strong solutions for every $\nu > 0$, we want to show that, as $\nu \to \nu_0$ for some fixed $\nu_0 > 0$, the solutions to these equations converge to the unique strong solutions of the respective equations (in the sense of Definitions 3.2 and 3.3) of the formal sensitivity equations (3.1) and (3.2) with initial data $u_0 \equiv 0$ in Theorems 3.9 and 4.1. We additionally prove that weak solutions exist for the sensitivity equations (3.1) with initial data $u_0 \equiv 0$ in Theorem 3.7. Via this method of proof, we show that the solution $\widetilde{u}$ is a Fréchet derivative in the sense of $L^2(0,T;H)$. Specifically, for weak solutions, $\nu \in \mathbb{R}^+ \mapsto u \in L^2(0,T;H)$, and hence the *Fréchet derivative in* $L^2(0,T;H)$ will be defined as $\mathcal{A} : \mathbb{R} \to L^2(0,T;H)$ such that $\mathcal{A}(\delta) \mapsto \delta\widetilde{u}$. Analogously, for strong solutions, $\nu \in \mathbb{R}^+ \mapsto u \in L^2(0,T;V)$, and hence the *Fréchet derivative in* $L^2(0,T;V)$ will be defined as $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$ such that $\mathcal{A}(\delta) \mapsto \delta\widetilde{u}$.

Theorem 3.7. *Let $T > 0$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence such that $\nu_n \to \nu$ as $n \to \infty$. Let*

- *u be a solution to* (2.1) *with viscosity ν, forcing $f \in L^\infty(0,\infty;H)$, and initial data $u_0 \in V$;*

- *u_n solve (2.1) with viscosity ν_n, forcing $f \in L^\infty(0,\infty;H)$, and initial data $u_0 \in V$;*
- *$\{w_n\}_{n\in\mathbb{N}}$ be a sequence of strong solutions to (3.4) with $w_n(0) = 0$ with corresponding viscosity ν_n and solutions to (2.1) u_n and u.*

Then there exists a subsequence of $\{w_n\}_{n\in\mathbb{N}}$ that converges in $L^2(0,T;H)$ to a unique weak solution $\widetilde{u}$ of (3.1) with initial data $u_0 \equiv 0$. Furthermore, the operator $\mathcal{A} : \mathbb{R} \to L^2(0,T;H)$ given by $\mathcal{A}(\delta) = \widetilde{u}\delta$ is the Fréchet derivative in the sense of $L^2(0,T;H)$ of the solution u with respect to ν.

Remark 3.8. Theorem 3.7 holds for more general initial data via a similar proof under certain assumptions on the initial data; indeed, assuming that the initial data of (2.1) has a derivative with respect to the viscosity and that it lies in an appropriate space and is a limit of difference quotients of the initial data corresponding to the sequence of solutions of (2.1), one can show a similar result. However, note that the sensitivity equations are a model for the evolution of the instantaneous change in a solution with respect to changes in the viscosity, hence the natural initial condition to consider is the case of identically-zero initial data. Thus, to avoid obfuscation, we work in the natural setting of identically-zero initial data; however, we provide a definition of weak and strong solutions for general initial data above. Similar remarks holds for all theorems below.

Proof. Let $T > 0$ be given. Let N sufficiently large such that for all $n > N$, $\{\nu_n\}_{n\in\mathbb{N}} \subset (\frac{\nu}{2}, \frac{3\nu}{2})$. Then, one can follow the proof of strong solutions for (2.1) as in, e.g., [18, 31, 65, 70], to obtain bounds on $\{u_n\}$ for $n > N$ in the appropriate spaces that are independent of ν_n:

$$\|u_n\|^2_{L^\infty(0,T;V)} \le \|u_n(0)\|^2 + \frac{\|f\|^2_{L^2(0,T;H)}}{\nu_n} \le \|u_0\|^2 + \frac{2\|f\|^2_{L^2(0,T;H)}}{\nu}$$

and

$$\|u_n\|^2_{L^2(0,T;\mathcal{D}(A))} \le \frac{1}{\nu_n}\|u_n(0)\|^2 + \frac{\|f\|^2_{L^2(0,T;H)}}{\nu_n^2} \le \frac{2}{\nu}\|u_0\|^2 + \frac{4\|f\|^2_{L^2(0,T;H)}}{\nu^2}.$$

Note that $\|f\|^2_{L^2(0,T;H)} < \infty$ since all bounded functions are locally integrable. Hence there is a subsequence that is relabeled $u_n \to u_1$ in $L^2(0,T;V)$ for some function u_1. Continuing to follow the proof of strong solutions for (2.1) as in e.g. [18, 31, 65, 70], we note that $\frac{du_n}{dt}$ is uniformly bounded in n in $L^2(0,T;H)$. Hence, we can find a subsequence which we relabel $\{u_n\}$ such that

$$\begin{aligned} \frac{du_n}{dt} &\rightharpoonup \frac{du_1}{dt} && \text{in } L^2(0,T;H) \\ \nu_n A u_n &\rightharpoonup \nu A u_1 && \text{in } L^2(0,T;H) \\ B(u_n,u_n) &\rightharpoonup B(u_1,u_1) && \text{in } L^2(0,T;H). \end{aligned}$$

Indeed, u_1 satisfies (2.1) with corresponding viscosity ν and thus, by uniqueness and the fact that $u_n \to u_1$ in $L^2(0,T;V)$, it follows that $u_1 = u$.

Let w_n be the strong solution to (3.4) with $\nu = \nu_n$. Taking the action of (3.4) on w_n and using (2.7), (2.5), Lemma 2.2, and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}|w_n|^2 + \nu_n\|w_n\|^2 \le \frac{c^2}{\nu_n}\|u_1\|^2|w_n|^2 + \frac{\nu_n}{4}\|w_n\|^2 + \frac{1}{2\nu_n}\|u_1\|^2 + \frac{\nu_n}{2}\|w_n\|^2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}|w_n|^2 + \frac{\nu_n}{4}\|w_n\|^2 \le \frac{c^2}{\nu_n}\|u\|^2|w_n|^2 + \frac{1}{2\nu_n}\|u\|^2. \tag{3.10}$$

Dropping the second term on the left hand side, we obtain

$$\frac{1}{2}\frac{d}{dt}|w_n|^2 \le \frac{c^2}{\nu_n}\|u\|^2|w_n|^2 + \frac{1}{2\nu_n}\|u\|^2.$$

Taking the integral with respect to time on $[0,T]$ and applying Grönwall's inequality, then for a.e. $t\in[0,T]$,

$$\begin{aligned}|w_n(t)|^2 &\le \Big[\frac{1}{\nu_n}\int_0^T \|u\|^2dt\Big]\exp\Big(\int_0^T \frac{2c^2}{\nu_n}\|u\|^2dt\Big)\\ &\le \Big[\frac{2}{\nu}\int_0^T \|u\|^2dt\Big]\exp\Big(\int_0^T \frac{4c^2}{\nu}\|u\|^2dt\Big) =: K_1.\end{aligned}$$

Since $u\in L^2(0,T;V)$, then w_n is bounded uniformly in $L^\infty(0,T;H)$.

Next, refraining from dropping the second term on the left hand side of (3.10), we estimate

$$\begin{aligned}\frac{\nu_n}{4}\int_0^T \|w_n\|^2dt &\le \frac{c^2}{\nu_n}\int_0^T \|u\|^2|w_n|^2dt + \frac{1}{2\nu_n}\int_0^T \|u\|^2dt\\ &\le K_1\frac{c^2}{\nu_n}\int_0^T \|u\|^2dt + \frac{1}{2\nu_n}\int_0^T \|u\|^2dt\end{aligned}$$

Rewriting, we obtain

$$\begin{aligned}\int_0^T \|w_n\|^2dt &\le K_1\frac{4c^2}{\nu_n^2}\int_0^T \|u\|^2dt + \frac{2}{\nu_n^2}\int_0^T \|u\|^2dt\\ &\le K_1\frac{16c^2}{\nu^2}\int_0^T \|u\|^2dt + \frac{8}{\nu^2}\int_0^T \|u\|^2dt\end{aligned}$$

Thus, w_n is bounded above uniformly in $L^2(0,T;V)$ with respect to n. Hence, by the Banach-Alaoglu Theorem, there exists a subsequence, relabeled as (w_n), such that

$$w_n \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0,T;H) \quad\text{and}\quad w_n \rightharpoonup w \text{ in } L^2(0,T;V). \tag{3.11}$$

Using (3.11), note that all uniform bounds in n on the terms in (3.4) in $L^2(0,T;V')$ are obtained in a similar manner to the proof of weak solutions for (2.1) except for the term $B(u_n,w_n)$. However, by Lemma 2.1,

$$\|B(u_n,w_n)\|_{L^2(0,T;V')} \le k\|u_n\|_{L^\infty(0,T;H)}\|w_n\|_{L^\infty(0,T;H)}\|u_n\|_{L^2(0,T;V)}\|w_n\|_{L^2(0,T;V)},$$

and due to the following standard bounds on u_n (which can be found in [18, 31, 65, 70], etc.) and the fact that $\nu_n\in(\frac{\nu}{2},\frac{3\nu}{2})$,

$$\|u_n\|^2_{L^\infty(0,T;H)} \le |u_0^n|^2 + \frac{\|f\|_{L^\infty(0,T;H)}}{\lambda_1^2\nu_n^2} \le |u_0|^2 + \frac{4\|f\|_{L^\infty(0,T;H)}}{\lambda_1^2\nu^2}$$

and

$$\|u_n\|^2_{L^2(0,T;V)} \le \frac{1}{\nu_n}|u^n(0)|^2 + \frac{\|f\|^2_{L^\infty(0,T;H)}}{\lambda_1\nu_n^2}T \le \frac{2}{\nu}|u_0|^2 + \frac{4\|f\|^2_{L^\infty(0,T;H)}}{\lambda_1\nu^2}T,$$

and thus $\|B(u_n,w_n)\|_{L^2(0,T;V')}$ is bounded above uniformly in n independent of ν_n. Hence, independent of ν_n, dw_n/dt is bounded uniformly in n. Thus, by the Aubin Compactness Theorem, $w_n\to w$ strongly in $L^2(0,T;H)$. Utilizing these bounds and the convergence properties, we can follow the usual arguments to state there exists a weak solution in the sense of Definition 3.2, where weak continuity can be proven directly in the same manner as for weak solutions to (2.12) and strong continuity follows from Lemma 2.2. Furthermore, weak continuity in H follows due to the bounds on each of the terms above. The initial condition is satisfied by construction. To prove uniqueness, suppose that there exist two weak solutions $\widetilde{u}_1$ and $\widetilde{u}_2$. We consider the difference of the equations

$$\frac{d}{dt}\widetilde{u}_1 + B(\widetilde{u}_1,u) + B(u,\widetilde{u}_1) + \nu A\widetilde{u}_1 + Au_1 = 0$$

and

$$\frac{d}{dt}\widetilde{u}_2 + B(\widetilde{u}_2,u) + B(u,\widetilde{u}_2) + \nu A\widetilde{u}_2 + Au_1 = 0,$$

which, defining $U := \widetilde{u}_1 - \widetilde{u}_2$, yields

$$\frac{d}{dt}U + B(U,u) + B(u,U) + \nu AU = 0$$

with $U(0) = 0$. Taking the action on U, using (2.5), and applying the Lions-Magenes Lemma 2.2,

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \langle B(U,u), U\rangle + \nu\|U\|^2 = 0.$$

Thus, by (2.7) and Young's inequality,

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \nu\|U\|^2 \leq c\|U\||U|\|u\| \leq \frac{c^2}{\nu}\|u\|^2|U|^2 + \frac{\nu}{2}\|U\|^2.$$

Dropping the second term and applying Grönwall's inequality, for a.e. $0 \leq t \leq T$,

$$|U(t)|^2 \leq |U(0)|^2 \exp\Big(\int_0^T \frac{c^2}{2\nu}\|u\|^2 dt\Big).$$

Thus, $\exp\Big(\int_0^T \frac{c^2}{\nu}\|u\|^2 dt\Big) < \infty$ for all $T > 0$ and $U(0) = 0$. Hence, $\|U\|_{L^\infty(0,T;H)} = 0$, which implies that $U \equiv 0$. Hence, weak solutions to (3.4) are unique.

Finally, we want to show that the sequence of difference quotients defines a Fréchet derivative. Let $\mathcal{A} : \mathbb{R} \to L^2(0,T;H)$, $\mathcal{A}(\delta) \equiv \delta\widetilde{u}$. Note that $\mathcal{A}$ is a bounded linear operator since $\widetilde{u}$ is bounded in $L^2(0,T;H)$. Let $\delta_n = \nu_n - \nu$, so that $\delta_n \to 0$ as $n \to \infty$. Note that for any sequence $\nu_n \to \nu$, there is a subsequence such that $w_{n_k} \to \widetilde{u}$ in $L^2(0,T;H)$. However, by the fact that for every subsequence of $\{w_n\}_{n\in\mathbb{N}}$ we can find a convergent subsequence, we can conclude that $w_n \to \widetilde{u}$ in $L^2(0,T;H)$. We rewrite $w_n \to \widetilde{u}$ in $L^2(0,T;H)$ as

$$\left\|\frac{u(\nu+\delta_n) - u(\nu)}{\delta_n} - \widetilde{u}\right\|_{L^2(0,T;H)} = \frac{1}{|\delta_n|}\left\|u(\nu_1+\delta_n) - u(\nu_1) - \delta_n\widetilde{u}\right\|_{L^2(0,T;H)} \to 0.$$

Therefore, $\mathcal{A}$ defines a Fréchet derivative from $\mathbb{R}$ to $L^2(0,T;H)$. □

Theorem 3.9. *Let $T > 0$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence such that $\nu_n \to \nu$ as $n \to \infty$. Let*

- *u be the solution to (2.1) with viscosity ν, forcing $f \in L^\infty(0,\infty;H)$, and initial data u_0;*
- *u_n solve (2.1) with viscosity ν_n, forcing $f \in L^\infty(0,\infty;H)$, and initial data $u_0 \in V$*
- *$\{w_n\}_{n\in\mathbb{N}}$ be a sequence of strong solutions to (3.4) with $w_n(0) = 0$ with corresponding viscosity ν_n and solutions to (2.1) u_n and u.*

Then there exists a subsequence of $\{w_n\}_{n\in\mathbb{N}}$ that converges in $L^2(0,T;V)$ to a unique strong solution $\widetilde{u}$ of (3.1) with initial data $u_0 \equiv 0$. Furthermore, the operator $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$ given by $\mathcal{A}(\delta) = \widetilde{u}\delta$ is the Fréchet derivative in $L^2(0,T;V)$ of the solution u with respect to ν.

Proof. Let $T > 0$ be given, and let $N > 0$ be large enough that $n > N$ implies $\{\nu_n\} \subset (\frac{\nu_1}{2}, \frac{3\nu_1}{2})$. Then by the argument in Theorem 3.7, we can obtain a subsequence which we relabel $\{u_n\}$ such that $u_n \to u$ in $L^2(0,T;V)$.

Consider w_n to be the strong solution to (3.4) with viscosity ν_n. Taking a justified inner product of (3.4) with Aw_n,

$$\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \nu_n|Aw_n|^2 = -(B(w_n,u), Aw_n) - (B(u_n,w_n), Aw_n) - (Au, Aw_n).$$

Applying Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \frac{\nu_n}{2}|Aw_n|^2 \leq -(B(w_n,u), Aw_n) - (B(u_n,w_n), Aw_n) + \frac{1}{2\nu_n}|Au|^2.$$

Applying (2.6) to the second bilinear term,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \frac{\nu_n}{2}|Aw_n|^2 &\leq -(B(w_n,u),Aw_n) + \|u_n\|_{L^\infty(\Omega)}\|w_n\||Aw_n| + \frac{1}{2\nu_n}|Au|^2 \\ &\leq \frac{2k^2}{\nu_n}|u_n||Au_n|\|w_n\|^2 + \frac{\nu_n}{8}|Aw_n|^2 \\ &\quad - (B(w_n,u),Aw_n) + \frac{1}{2\nu_n}|Au|^2\end{aligned}$$

and applying (2.8) to the first bilinear term,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \frac{3\nu_n}{8}|Aw_n|^2 &\leq \frac{2k^2}{\nu_n}|u_n||Au_n|\|w_n\|^2 + c|w_n|^{1/2}\|w_n\|^{1/2}\|u\|^{1/2}|Au|^{1/2}|Aw_n| \\ &\quad + \frac{1}{2\nu_n}|Au|^2 \\ &\leq \frac{2k^2}{\nu_n}|u_n||Au_n|\|w_n\|^2 + \frac{2c^2}{\lambda_1\nu_n}\|w_n\|^2\|u\||Au| \\ &\quad + \frac{\nu_n}{8}|Aw_n|^2 + \frac{1}{2\nu_n}|Au|^2\end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}\|w_n\|^2 + \frac{\nu_n}{2}|Aw_n|^2 \leq \Big(\frac{4k^2}{\nu_n}|u_n||Au_n| + \frac{4c^2}{\lambda_1\nu_n}\|u\||Au|\Big)\|w_n\|^2 + \frac{1}{\nu_n}|Au|^2.$$

Integrating on both sides in time, with $0 \leq t \leq T$,

$$\begin{aligned}\|w_n(t)\|^2+\frac{\nu_n}{2}\int_0^t |Aw_n|^2 ds &\leq \frac{1}{\nu_n}\int_0^t |Au(s)|^2 ds \\ &\quad + \int_0^t \Big(\frac{4k^2}{\nu_n}|u_n(s)||Au_n(s)| + \frac{4c^2}{\lambda_1\nu_n}\|u(s)\||Au(s)|\Big)\|w_n(s)\|^2 ds\end{aligned}$$

Dropping the second term on the left hand side, we apply Grönwall's inequality to obtain

$$\begin{aligned}\|w_n(t)\|^2 &\leq \alpha_n(t)\exp\Big(\int_0^t \frac{4k^2}{\nu_n}|u_n(s)||Au_n(s)| + \frac{4c^2}{\lambda_1\nu_n}\|u(s)\||Au(s)|ds\Big) \\ &\leq \alpha(t)\exp\Big(\int_0^t \frac{8k^2}{\nu}|u_n(s)||Au_n(s)| + \frac{8c^2}{\lambda_1\nu}\|u(s)\||Au(s)|ds\Big).\end{aligned}$$

where $\alpha_n(t) := \frac{2}{\nu_n}\int_0^t |Au(s)|^2 ds \leq \alpha(t) := \frac{4}{\nu}\int_0^t |Au(s)|^2 ds$. Since

$$\int_0^T |Au_n|^2 ds \leq \|u_0\|^2 + \frac{\|f\|^2_{L^2(0,T;H)}}{\nu_n^2}$$

as proven in, e.g., [18, 65, 31, 70], then

$$\begin{aligned}\sup_{t\in[0,T]}\|w_n(t)\|^2 &\le \alpha(T)\exp\left(\frac{8k^2}{\lambda_1^2\nu}\int_0^T |Au_n|^2ds + \frac{8c^2}{\lambda_1\nu}\|u(s)\||Au(s)|ds\right)\\ &\le \alpha(T)\exp\left(\frac{8k^2}{\lambda_1^2\nu}\left[\|u_0\|^2 + \frac{\|f\|^2_{L^2(0,T;H)}}{\nu_n^2}\right]\right)\\ &\quad + \alpha(T)\exp\left(\int_0^T \frac{8c^2}{\lambda_1\nu}\|u(s)\||Au(s)|ds\right)\\ &\le \alpha(T)\exp\left(\frac{8k^2}{\lambda_1^2\nu}\left[\|u_0\|^2 + \frac{4\|f\|^2_{L^2(0,T;H)}}{\nu^2}\right]\right)\\ &\quad + \alpha(T)\exp\left(\int_0^T \frac{8c^2}{\lambda_1\nu}\|u(s)\||Au(s)|ds\right)\end{aligned}$$

This implies that $w_n \in L^\infty(0,T;V)$ and $\{w_n\}$ is uniformly bounded in this space.

Additionally, considering again the inequality

$$\begin{aligned}\|w_n(t)\|^2+\frac{\nu_n}{2}\int_0^t |Aw_n|^2ds &\le \frac{1}{\nu_n}\int_0^t |Au(s)|^2ds\\ &+\int_0^t \Big(\frac{4k^2}{\nu_n}|u_n(s)||Au_n(s)| + \frac{4c^2}{\lambda_1\nu_n}\|u(s)\||Au(s)|\Big)\|w_n(s)\|^2ds.\end{aligned}$$

we set $t = T$, drop the first term on the left hand side, and bound the viscosity above to obtain

$$\begin{aligned}\int_0^T |Aw_n|^2ds \le \frac{8}{\nu^2}&\left(\int_0^T |Au(s)|^2ds\right)\\ &+\int_0^T \Big(\frac{32k^2}{\lambda_1\nu^2}|Au_n(s)|^2 + \frac{32c^2}{\lambda_1\nu^2}\|u(s)\||Au(s)|\Big)\|w_n(s)\|^2ds\end{aligned}$$

By the fact that $\{\|u_n\|_{L^2(0,T;\mathcal{D}(A))}\}$ is bounded above in n, as demonstrated in Theorem 3.7, and the result that $\{\|w_n\|_{L^\infty(0,T;V)}\}$ is bounded above uniformly in n, it follows that $\{\|w_n\|_{L^2(0,T;\mathcal{D}(A))}\}$ is bounded above uniformly in n. Since $\{w_n\}$ is bounded above uniformly in n in both $L^\infty(0,T;V)$ and $L^2(0,T;\mathcal{D}(A))$, then we can conclude that there exists a subsequence, which we relabel as $\{w_n\}$, such that

$$w_n \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0,T;V) \text{ and } w_n \rightharpoonup w \text{ in } L^2(0,T;\mathcal{D}(A)). \tag{3.12}$$

Using (3.12), note that all uniform bounds in n on the terms in (3.4) in $L^2(0,T;H)$ are obtained in a similar manner to the proof of strong solutions for the (2.1) and are independent of ν_n except for the bilinear terms. The bilinear terms are bounded uniformly in $L^2(0,T;H)$ with respect to n, due to Lemma 2.1. Hence, $\frac{dw_n}{dt}$ is bounded above uniformly in n in $L^2(0,T;H)$. Thus, by the Aubin Compactness Theorem, $w_n \to w$ strongly in $L^2(0,T;V)$. Utilizing these bounds and convergence rates, it is classical to show that w is a strong solution in the sense of Definition 3.2. Specifically, as in, e.g., [65, 70, 31, 18], $w \in C([0,T];V)$. The initial condition is also satisfied by construction. Uniqueness holds due to the results in Theorem 3.7.

Finally, following a similar argument as in the proof of Theorem 3.7, one can show that the sequence of difference quotients gives rise to a Fréchet derivative $\mathcal{A}(\delta) \equiv \delta\widetilde{u}$, where $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$. Let $\mathcal{A}(\delta) = \delta\widetilde{u}$. Let $\delta_n = \nu_n - \nu$, so that $\delta_n \to 0$ as $n \to \infty$. Note that for any sequence $\nu_n \to \nu$, there is a subsequence such that $w_{n_k} \to \widetilde{u}$ in $L^2(0,T;V)$. However, by the fact that for every subsequence of $\{w_n\}_{n\in\mathbb{N}}$ we can find a convergence subsequence, one can in fact conclude that the entire sequence $w_n \to \widetilde{u}$ in $L^2(0,T;V)$. We rewrite $w_n \to \widetilde{u}$ in $L^2(0,T;V)$ as

$$\left\| \frac{u(\nu_1+\delta_n)-u(\nu_1)}{\delta_n} - \widetilde{u} \right\|_{L^2(0,T;V)} = \frac{1}{|\delta_n|}\left\| u(\nu_1+\delta_n) - u(\nu_1) - \delta_n \widetilde{u} \right\|_{L^2(0,T;V)} \to 0.$$

□

Remark 3.10. Ideally, one would like to show that the difference quotients of weak solutions give rise to a Fréchet derivative in the sense of the Leray space $L^2(0,T;V)\cap L^\infty(0,T;H)$, rather than just the space $L^2(0,T;H)$. Unfortunately, with the given hypotheses the method of proof employed here does not seem to allow for a proof in this context, since we only have weak or weak-$*$ convergence in the relevant spaces. It may be possible to prove such a result, using, e.g., the methods described in [66], but such a proof would be lengthy and distract from the main focus of the present work. Hence, we plan to explore these details in a future work.

Finally we consider the sensitivity of the Euler equations to a viscous regularization. This leads to the following statement of the formal sensitivity equations:

$$\widetilde{u}_t + \widetilde{u}\cdot\nabla u + u\cdot\nabla\widetilde{u} - \triangle u + \nabla\widetilde{p} = 0, \tag{3.13a}$$

$$\nabla\cdot\widetilde{u} = 0, \tag{3.13b}$$

where (u,p) is the solution to the Euler equations. With (u_ν, p_ν) is the solution to (2.1) with viscosity ν, the corresponding system of difference quotient equations is

$$w_t + u_\nu\cdot\nabla w + w\cdot\nabla u - \triangle u + \nabla P = 0, \tag{3.14a}$$

$$\nabla\cdot w = 0, \tag{3.14b}$$

$$w(x,0) = 0, \tag{3.14c}$$

where $w = \frac{u_\nu - u}{\nu}$ and $P = \frac{p_\nu - p}{\nu}$.

We only utilize the least amount of regularity required on the initial data in Theorem 2.5 to preserve the convergence of strong solutions to the corresponding systems of difference quotient equations in the same spaces as in Theorem 3.9. We state the definition of strong solutions to (3.13) in the same sense as for (3.5) to illustrate the minimal needed additional regularity for the convergence of solutions to (3.14) to those of (3.13).

Definition 3.11. Let $T>0$, and $s>2$. Let $u\in C([0,T];\mathcal{D}(A^{s/2}))$ be a strong solution to (2.21) with initial data $u_0\in\mathcal{D}(A^{s/2})$ and $f\equiv 0$. A *strong solution* of (3.13) is an element $\widetilde{u}\in L^2(0,T;\mathcal{D}(A))\cap C([0,T];V)$ satisfying $\frac{d\widetilde{u}}{dt}\in L^2(0,T;H)$ and

$$\left\langle \frac{d}{dt}\widetilde{u},\phi \right\rangle + \langle B(\widetilde{u},u),\phi\rangle + \langle B(u,\widetilde{u}),\phi\rangle + \langle Au,\phi\rangle = 0 \tag{3.15}$$

for a.e. $t\in[0,T]$, for all $\phi\in H$, and initial data $\widetilde{u}_0\in\mathcal{D}(A^{s/2})$. Moreover, the map $\widetilde{u_0}\in H\mapsto\widetilde{u}\in H$ is continuous for a.e. $t\in[0,T]$, i.e. the solution depends continuously on the initial data.

Remark 3.12. We do not extend this analysis to the equations involving the data assimilation algorithm, as the Euler equations are not dissipative.

Definition 3.13. Let $T>0$, and $s>2$, and suppose $u,u_\nu\in L^2(0,T;\mathcal{D}(A))\cap C([0,T];V)$. A strong solution to (3.14) is an element $w\in L^2(0,T;\mathcal{D}(A))\cap C([0,T];V)$, such that $\frac{d}{dt}w\in L^2(0,T;H)$, and for all $\phi\in H$ and for a.e., $t\in[0,T]$,

$$\left(\frac{d}{dt}w,\phi\right) + (B(w,u),\phi) + (B(u_\nu,w),\phi) + (Au,\phi) = 0.$$

Moreover, the map $u_0\in H\mapsto u\in H$ is continuous for a.e. $t\in[0,T]$, i.e. the solution depends continuously on the initial data.

Lemma 3.14. *Let $T > 0$ be given, and let $u, u_1 \in L^2(0,T;\mathcal{D}(A)) \cap C([0,T];V)$ be strong solutions to* (2.1) *and* (2.21)*, respectively. Then there exists a unique strong solution to* (3.14)*.*

Proof. Since $\triangle u \in L^2(0,T;H)$, and u_ν is sufficiently smooth, existence follows exactly as in the standard theory for the 2D Navier-Stokes equations. Uniqueness is also similar, but we carry out the short proof to show the dependencies on various quantities. Hence, suppose w_1 and w_2 are strong solutions to (3.14). Let $U := w_1 - w_2$, so that

$$\frac{d}{dt}U + B(U,u) + B(u_1,U) = 0$$

with $U(0) = 0$ in V. Taking the H-inner-product with U,

$$\frac{1}{2}\frac{d}{dt}|U|^2 = -(B(U,u),U) \leq \|\nabla u\|_{L^\infty}|U|^2,$$

and applying Grönwall's inequality yields

$$|U(t)|^2 \leq |U(0)|^2 e^{2\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds}. \tag{3.16}$$

Since $u \in C([0,T],\mathcal{D}(A^{s/2}))$, $s > 2$, and $U(0) = 0$, we necessarily have uniqueness. Furthermore, note that (3.16) implies that the map $u_0 \in H \mapsto u \in H$ is continuous. □

Remark 3.15. In the following theorem, we note that it is worth noting that the analysis can be carried out in a similar manner as when the viscosity is non-zero for the formal sensitivity equations, as one does not encounter the difficulties expected in energy estimates for Euler. The reason for this is that the formal derivative of the viscous regularization results in a diffusive term that is independent of the viscosity.

Theorem 3.16. *Let $T > 0$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence such that $\nu_n \to 0$ as $n \to \infty$. Let*

- *u be the solution to* (2.21) *with forcing $f \equiv 0$ and initial data $u_0 \in \mathcal{D}(A^{s/2})$, $s > 2$,*
- *u_n solves* (2.1) *with viscosity ν_n, forcing $f \equiv 0$, and initial data $u_0 \in \mathcal{D}(A^{s/2})$, $s > 2$,*
- *$\{w_n\}$ be a sequence of strong solutions to* (3.14) *with corresponding viscosity ν_n, solution to* (2.1) *u_n, and solution to* (2.22) *u.*

Then $\{w_n\}_{n\in\mathbb{N}}$ that converges in $L^2(0,T;V)$ to a unique strong solution $\widetilde{u}$ of (3.13) *with initial data $u_0 \equiv 0$. Furthermore, the operator $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$ given by $\mathcal{A}(\delta) = \widetilde{u}\delta$ is the Fréchet derivative in $L^2(0,T;V)$ of the weak solution u with respect to ν.*

Proof. Taking the difference between (2.1) and (2.21), and dividing by the viscosity ν_n, we obtain the system of difference quotient equations, we obtain

$$\frac{d}{dt}w_n + B(w_n,u) + B(u_n,w_n) + Aw_n + Au = 0.$$

Taking the inner product with Aw_n then applying (2.7) and Young's inequality,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + |Aw_n|^2 &= -(B(w_n,u),Aw_n) - (B(u_n,w_n),Aw_n) - (Au,Aw_n) \\
&\leq c|w_n|^{1/2}\|w_n\|^{1/2}\|u\|^{1/2}|Au|^{1/2}|Aw_n| \\
&\quad + c|u_n|^{1/2}\|u_n\|^{1/2}\|w_n\|^{1/2}|Aw_n|^{1/2}|Aw_n| + |Au||Aw_n| \\
&\leq \left(\frac{c^2}{\sqrt{\lambda_1}}\|u\||Au|\right)\|w_n\|^2 + \frac{1}{4}|Aw_n|^2 + \frac{(32c/3)^4}{4}|u_n|^2\|u_n\|^2\|w_n\|^2 + \frac{1}{8}|Aw_n|^2 \\
&\quad + 2|Au|^2 + \frac{1}{8}|Aw_n|^2,
\end{aligned}$$

which implies

$$\frac{d}{dt}\|w_n\|^2 + |Aw_n|^2 \leq \left(\frac{2c^2}{\sqrt{\lambda_1}}\|u\||Au|\right)\|w_n\|^2 + \frac{(32c/3)^4}{2}|u_n|^2\|u_n\|^2\|w_n\|^2 + 4|Au|^2.$$

Dropping the second term on the left hand side and applying Grönwall's inequality, for any $0 \leq t \leq T$,

$$\|w_n(t)\|^2 \leq 4\int_0^t |Au(s)|^2 ds \exp\left(\int_0^t \frac{2c^2}{\sqrt{\lambda_1}}\|u(\tau)\||Au(\tau)| + \frac{(32c/3)^4}{2}|u_n(\tau)|^2\|u_n(\tau)\|^2 d\tau\right), \tag{3.17}$$

which is bounded due to the fact that $u \in C([0,T];\mathcal{D}(A^{s/2}))$ and u_n is bounded uniformly as well in $C([0,T];\mathcal{D}(A^{s/2}))$. Furthermore, retaining the second term above and integrating, we obtain

$$\begin{aligned}
&\int_0^t |Aw_n(s)|^2 \\
&\leq \int_0^t \left(\frac{2c^2}{\sqrt{\lambda_1}}\|u(s)\||Au(s)| + \frac{(32c/3)^4}{2}|u_n(s)|^2\|u_n(s)\|^2\|w_n(s)\|^2\right) ds + 4\int_0^t |Au(s)|^2 ds \\
&\leq \|w_n(s)\|^2_{L^\infty(0,T;V)}\int_0^t \left(\frac{2c^2}{\sqrt{\lambda_1}}\|u(s)\||Au(s)| + \frac{(32c/3)^4}{2}|u_n(s)|^2\|u_n(s)\|^2\right) ds + 4\int_0^t |Au(s)|^2 ds,
\end{aligned}$$

which is bounded by (3.17). By Theorem 2.5 we have that $u \in L^2(0,T;\mathcal{D}(A))$ and that u_n is bounded uniformly in $C([0,T];V)\cap L^2(0,T;\mathcal{D}(A))$ with respect to n. The convergence results in Theorem 2.5 can also be applied such that following the exact same arguments as in Theorem 3.9, we obtain the existence of a strong solution to $\widetilde{u}$ as per Definition 3.13. Uniqueness and continuity with respect to the initial data also follows as in Lemma 3.14. Finally, defining $\mathcal{A}(\delta) \equiv \delta\widetilde{u}$, $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$, we also obtain a formal Fréchet derivative in the same manner as in Theorem 3.9. ☐

4. Extension to a Data Assimilation Algorithm

In this section, we extend our analysis above to the context of a data assimilation algorithm, as discussed in the introduction.

Theorem 4.1. *Let $T > 0$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence such that $\nu_n \to \nu_{appx}$, $\nu_{appx} > 0$, as $n \to \infty$. Choose μ and h such that $4\mu c_0 h^2 \leq \nu_n \leq \frac{3\nu_{appx}}{2}$. Let*

- *v be the solution to (2.16) with viscosity ν_{appx}, forcing $f \in L^\infty(0,\infty;H)$, and initial data v_0;*
- *$\widetilde{u}$ be a strong solution to (3.1);*
- *v_n solve (2.16) with viscosity ν_n, forcing $f \in L^\infty(0,\infty;H)$, and initial data $v_0 \in V$;*
- *$\{\widetilde{w}_n\}_{n\in\mathbb{N}}$ be a sequence of strong solutions to (3.5) with $\widetilde{w}_n(0) = 0$ with corresponding viscosity ν_n and solutions to (2.16) u_n and u.*

Then there exists a subsequence of $\{\widetilde{w}_n\}_{n\in\mathbb{N}}$ that converges in $L^2(0,T;V)$ to a unique solution $\widetilde{v}$ of (3.2) with initial data $u_0 \equiv 0$. Furthermore, the operator $\mathcal{A} : \mathbb{R} \to L^2(0,T;V)$ given by $\mathcal{A}(\delta) = \widetilde{u}\delta$ is the Fréchet derivative in $L^2(0,T;V)$ of the solution u with respect to ν_{appx}.

Proof. Let $T > 0$. Note that since $\{\nu_n\} \subset (\frac{\nu_{\text{appx}}}{2}, \frac{3\nu_{\text{appx}}}{2})$ for $n > N$ for some sufficiently large N, we can follow the proof of strong solutions for (2.16) in [4] to obtain bounds on $\{v_n\}_{n>N}$ in the appropriate spaces that are independent of ν_n. First, we note that [4] quickly proves $|f + \mu P_\sigma I_h(u_n)|^2 \leq M_n$ since $|P_\sigma I_h(u_n)|^2 \leq (c_0\sqrt{h} + \lambda_1^{-1/2})\|u_n\|^2$. However, since u_n is bounded above uniformly in n (see the proof of Theorem 3.7), we have that $|f + \mu P_\sigma I_h(u_n)|^2 \leq m$ for some m independent of n. Thus, we have the following bounds from [4] bounded above uniformly in n:

$$\|v_n\|^2_{L^\infty(0,T;H)} \leq |v_n(0)|^2 + \frac{m}{\mu\nu_n\lambda_1} \leq |v_0|^2 + \frac{2m}{\mu\nu_{\text{appx}}\lambda_1}, \tag{4.1}$$

$$\|v_n\|^2_{L^2(0,T;V)} \leq \frac{1}{\nu_n}|v_n(0)|^2 + \frac{T}{\mu\nu_n}m \leq \frac{2}{\nu_{\text{appx}}}|v_0|^2 + \frac{2T}{\mu\nu_{\text{appx}}}m, \tag{4.2}$$

and, defining

$$\exp\Big\{\frac{c}{(\nu_2)_n^3}\int_0^T |v_n|^2\|v_n\|^2 ds\Big\} \equiv \frac{1}{\psi^n_{\nu_n}(T)} \leq \frac{1}{\psi^n_{\nu_{\text{appx}}}(T)} \equiv \exp\Big\{\frac{8c}{\nu_{\text{appx}}^3}\int_0^T |v_n|^2\|v_n\|^2 ds\Big\},$$

which is bounded above uniformly in n due to (4.1) and (4.2),

$$\|v_n\|^2_{L^\infty(0,T;V)} \leq \frac{1}{\psi^n_{\nu_n}(T)}\Big[\|v_n(0)\|^2 + \frac{4T}{\nu_n}m\Big] \leq \frac{1}{\psi^n_{\nu_{\text{appx}}}(T)}\Big[\|v_0\|^2 + \frac{8T}{\nu_{\text{appx}}}m\Big]. \tag{4.3}$$

Furthermore,

$$\begin{aligned}\|v_n\|^2_{L^2(0,T;\mathcal{D}(A))} &\leq \frac{1}{\nu_n}\|v_n(0)\|^2 + \frac{c}{(\nu_2)^3_n}\int_0^T (|v_n|^2\|v_n\|^4 + \frac{4}{\nu_n}|f + P_\sigma I_h(u_n)|^2)ds \\ &\leq \frac{2}{\nu_{\text{appx}}}\|v_0\|^2 + \frac{8c}{\nu^3_{\text{appx}}}\int_0^T |v_n|^2\|v_n\|^4 ds + \frac{8T}{\nu_{\text{appx}}}m,\end{aligned}$$

which is bounded above uniformly in n due to (4.1), (4.2), (4.3). Hence, we will obtain a subsequence that is relabeled $v_n \to v_1$ in $L^2(0,T;V)$ for some function v_1. Indeed, we see that by identical arguments presented in Theorem 3.7, $v_1 = v$. Also due to Poincaré's inequality, we obtain that $v_n \to v$ in $L^2(0,T;H)$.

Let $\{\widetilde{w}_n\}_{n\in\mathbb{N}}$ be a sequence of solutions to (3.5). We consider the Leray projection of (3.5):

$$\frac{d}{dt}\widetilde{w}_n + B(\widetilde{w}_n, v) + B(v_n, \widetilde{w}_n) + \nu_n A\widetilde{w}_n + Av = \mu P_\sigma I_h(w_n - \widetilde{w}_n).$$

The existence proof for (3.2) closely follows the proof of Theorem 3.9, with some modifications on the bounds of $\widetilde{w}_n$ which we show below. Taking the inner product with $A\widetilde{w}_n$ and proceeding as in the proof of Theorem 3.9, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{w}_n\|^2 + \frac{\nu_n}{4}|A\widetilde{w}_n|^2 &\leq \Big(\frac{2k^2}{\nu_n}|v_n||Av_n| + \frac{2c^2}{\lambda_1\nu_n}\|v\||Av|\Big)\|\widetilde{w}_n\|^2 \\ &\quad + \frac{1}{2\nu_n}|Av|^2 + \mu(I_h(w_n - \widetilde{w}_n), A\widetilde{w}_n).\end{aligned} \tag{4.4}$$

We slightly modify the inequalities obtained in [4] for the interpolant term,

$$\begin{aligned}-\mu(I_h(\widetilde{w}_n), A\widetilde{w}_n) &\leq \frac{4\mu^2}{\nu_n}|\widetilde{w}_n - I_h(\widetilde{w}_n)|^2 + \frac{\nu_n}{16}|A\widetilde{w}_n|^2 - \mu\|\widetilde{w}_n\|^2 \\ &\leq \frac{4\mu^2 c_0 h^2}{\nu_n}\|\widetilde{w}_n\|^2 + \frac{\nu_n}{16}|A\widetilde{w}_n|^2 - \mu\|\widetilde{w}_n\|^2 \\ &\leq \frac{\nu_n}{16}|A\widetilde{w}_n|^2.\end{aligned}$$

Also,

$$\mu|(I_h(w_n), A\widetilde{w}_n)| \leq \frac{4\mu^2}{\nu_n}|w_n|^2 + \frac{\nu_n}{16}|A\widetilde{w}_n|^2.$$

Using these inequalities in (4.4):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{w}_n\|^2 + \frac{\nu_n}{8}|A\widetilde{w}_n|^2 &\leq \Big(\frac{2k^2}{\nu_n}|v_n||Av_n| + \frac{2c^2}{\lambda_1\nu_n}\|v\||Av|\Big)\|\widetilde{w}_n\|^2 \\ &\quad + \frac{1}{2\nu_n}|Av|^2 + \frac{4}{\nu_n}|w_n|^2 \\ &\leq \Big(\frac{2k^2}{\nu_n}|v_n||Av_n| + \frac{2c^2}{\lambda_1\nu_n}\|v\||Av|\Big)\|\widetilde{w}_n\|^2 \\ &\quad + \frac{1}{2\nu_n}|Av|^2 + \frac{4}{\lambda_1^2\nu_n}|Aw_n|^2.\end{aligned}$$

Following identical arguments as in Theorem 3.9 with

$$\alpha_n(t) := \frac{1}{2\nu_n}|Av_1|^2 + \frac{4}{\lambda_1^2\nu_n}|Aw_n|^2 \leq \alpha(t) := \frac{1}{\nu_{\text{appx}}}|Av|^2 + \frac{8}{\lambda_1^2\nu_{\text{appx}}}|Aw_n|^2,$$

along with the fact that $P_\sigma I_h(w_n - \widetilde{w}_n)$ is bounded uniformly in n in $L^2(0,T;H)$, we obtain a subsequence relabeled $\widetilde{w}_n \to \widetilde{w}$ in $L^2(0,T;V)$. Indeed, let $\phi \in L^2(0,T;H)$; then

$$\begin{aligned}
\int_0^T (P_\sigma I_h(w_n - \widetilde{w}_n) - P_\sigma I_h(w - \widetilde{w}), \phi)ds &\le \int_0^T |I_h(w_n - \widetilde{w}_n) - I_h(w - \widetilde{w})||\phi|ds \\
&\le \int_0^T |I_h(w_n - w) - I_h(\widetilde{w}_n - \widetilde{w})||\phi|ds \\
&\le \int_0^T |[(w_n - w) - (\widetilde{w}_n - \widetilde{w})] - I_h((w_n - w) - (\widetilde{w}_n - \widetilde{w}))||\phi|ds \\
&\quad + \int_0^T |(w_n - w) - (\widetilde{w}_n - \widetilde{w})||\phi|ds \\
&\le \sqrt{c_0}h \int_0^T \|(w_n - w) - (\widetilde{w}_n - \widetilde{w})\||\phi|ds + \frac{1}{\lambda_1^{1/2}} \int_0^T \|(w_n - w) - (\widetilde{w}_n - \widetilde{w})\||\phi|ds \\
&\le \sqrt{c_0}h \|w_n - w\|_{L^2(0,T;V)} \|\phi\|_{L^2(0,T;H)} + \frac{1}{\lambda_1^{1/2}} \|\widetilde{w}_n - \widetilde{w}\|_{L^2(0,T;V)} \|\phi\|_{L^2(0,T;H)}.
\end{aligned}$$

Additionally, since we now have that $\widetilde{w}_n \to w$ in $L^2(0,T;V)$, then $P_\sigma I_h(w_n - \widetilde{w}_n) \rightharpoonup P_\sigma I_h(w - \widetilde{w})$ in $L^2(0,T;H)$ and we conclude $\widetilde{w}$ is a strong solution in the sense of Definition 3.3.

To show that the solutions are unique, we consider the difference of the equations

$$\frac{d}{dt}\widetilde{v}_1 + B(\widetilde{v}_1, v) + B(v, \widetilde{v}_1) + \nu_{\text{appx}} A\widetilde{v}_1 + Av = \mu P_\sigma I_h(\widetilde{u} - \widetilde{v}_1)$$

and

$$\frac{d}{dt}\widetilde{v}_2 + B(\widetilde{v}_2, v) + B(v, \widetilde{v}_2) + \nu_{\text{appx}} A\widetilde{v}_2 + Av = \mu P_\sigma I_h(\widetilde{u} - \widetilde{v}_2)$$

which, defining $W := \widetilde{v}_1 - \widetilde{v}_2$, yields

$$\frac{d}{dt}W + B(W, v) + B(v, W) + \nu_{\text{appx}} AW = -\mu P_\sigma I_h(W)$$

with $W(0) = 0$. So, W must be a solution to the above equation. Taking the action on W and applying the Lions-Magenes Lemma 2.2,

$$\frac{1}{2}\frac{d}{dt}|W|^2 + \langle B(W, v), W\rangle + \nu_{\text{appx}}\|W\|^2 = \langle -\mu P_\sigma I_h(W), W\rangle$$

which implies that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|W|^2 + \nu_{\text{appx}}\|W\|^2 &\le c\|v\||W|\|W\| + \mu(\sqrt{c_0}h + \lambda_1^{-1})\|W\||W| \\
&\le \frac{\mu^2(\sqrt{c_0}h + \lambda_1^{-1})^2}{\nu_{\text{appx}}}|W|^2 + \frac{\nu}{4}\|W\|^2 \\
&\quad + \frac{c^2}{2\nu_{\text{appx}}}\|v\|^2|W|^2 + \frac{\nu_{\text{appx}}}{2}\|W\|^2.
\end{aligned}$$

Thus,

$$\frac{d}{dt}|W|^2 \le \Big(\frac{\mu^2(\sqrt{c_0}h + \lambda_1^{-1})^2}{\nu_{\text{appx}}} + \frac{c^2}{2\nu_{\text{appx}}}\|v\|^2\Big)|W|^2$$

and Grönwall's inequality implies, for a.e. $0 \le t \le T$,

$$|W(t)|^2 \le |W(0)|^2 \exp\Big(\int_0^T \frac{\mu^2(\sqrt{c_0}h + \lambda_1^{-1})^2}{\nu_{\text{appx}}} + \frac{c^2}{2\nu_{\text{appx}}}\|v\|^2 dt\Big).$$

But $W(0) = 0$, and thus $\|W\|_{L^\infty(0,T;H)} = 0$ implies that $W \equiv 0$. Hence, solutions to (3.5) are unique.

Finally, we want to show that the sequence of difference quotients defines a Fréchet derivative. Following a similar argument as in Theorem 3.7, the Fréchet derivative $\mathcal{A}(\delta) \equiv \delta\widetilde{u}$ maps $\mathbb{R} \to L^2(0,T;V)$. $\square$

5. Conclusion

In this article, we proved well-posedness of the sensitivity equations for the 2D incompressible Navier-Stokes equations and the associated AOT data assimilation system, as well as a viscous regularization of Euler. Specifically, we proved the existence and uniqueness of global solutions to these equations. A byproduct of the proof is that the sensitivity of solutions to the equations involved in the algorithm are bounded in appropriate spaces. Hence, changing the viscosity, or equivalently the Reynolds number, mid-simulation as in [14] does not result in major aberrations in the solution. We note that in the present context, our proof is somewhat non-standard, in that we proved the existence by showing that the difference quotients converge to a solution of the equations. We believe this is the first such rigorous proof that the sensitivity equations for the 2D Navier-Stokes equations are globally well-posed (even in the inviscid limit), although formal proofs have been given in other works, cited above.

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